$\wp(z, .34+.87i)$

$\ln(z), \sin(z), \cosh(z), 1/\cosh(z)$

$z, z^2, z^5, z^{\frac{1}{2}}, z^{\frac{1}{12}}$

$e^{1/z}$

$\Gamma(s), \zeta(s)$

Complex Analysis:

Appendix of Color-Plots

Alexander Atanasov

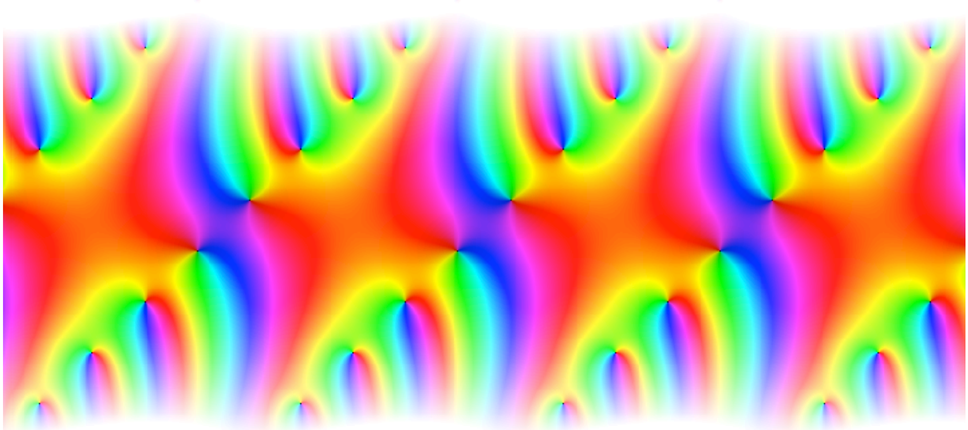
$\frac{1}{1+z^2}$

$\sqrt{1-\zeta^2}$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$\Theta(0|\tau), \Theta(z|i), \Theta\left(z\middle|\frac{1}{2}+\frac{1}{2}i\right), \Theta(z|.34+.87i), \Theta(z|1.2+.2i)$

$\mathrm{sn}(z)$



The purpose of this appendix is to show the reader of the associated work, "Complex Analysis: In Dialogue" how to envision complex functions as color mappings. Because we lack the ability to envision the four-dimensional structures of complex functions in their entirety, such color plots can replace the intuition that graphs of functions of a real variable once provided for us in the days of single and multivariable calculus.

It would have been possible to print a complex analysis book without any mention of color plots, and color plots may not prove instructive to every student. Still, I am driven by my own deep love for this type of representation to add this appendix as a humble supplement to my work.

The appendix is not strictly necessary in order to comprehend and interpret the passages in the main work. The reason for including color-plots is merely to share with the reader my own method of visualizing complex functions.
This specific method of visualization will prove very powerful indeed when it comes to seeing complex-valued polynomials, essential singularities, and Theta functions.

For love of simplicity, I shall write no more.

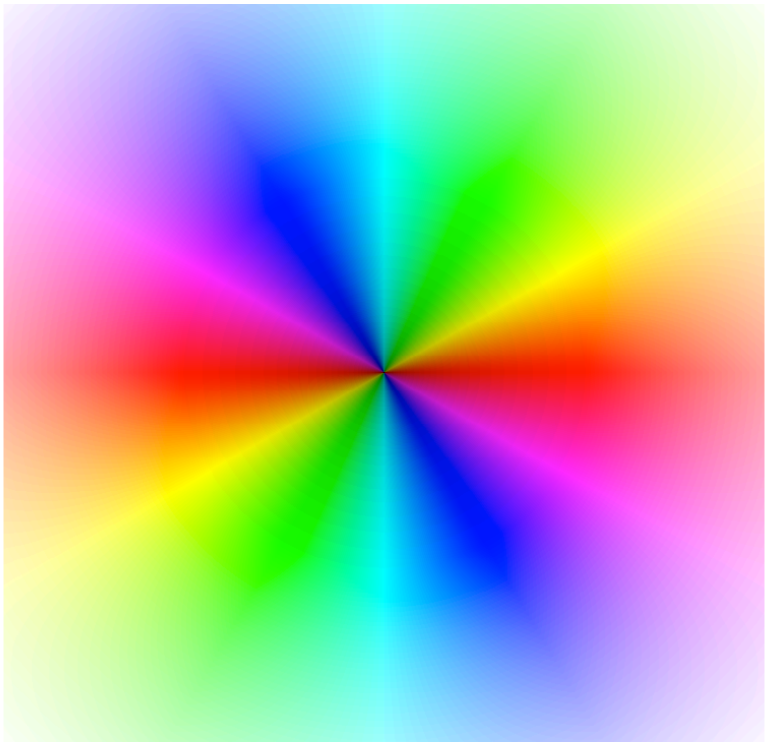
Appendix:

Image 1, $w = z$

This first image is the identity mapping on the complex plane and shows what points on the complex plane map to what colors. The frame of this image is $[-3.75, 3.75]$ on the real and imaginary axes. Clearly, the number zero will become black, the positive numbers will become red, and the negative numbers will become cyan. The positive imaginary numbers become greenish-yellow, and the negative imaginary numbers become violet. At large magnitudes, the numbers will begin to become white.

Saying "the plot range is $[-3.75, 3.75]$ for both components" means that the plot will show the values of f evaluated at all numbers that have real and imaginary components within that range. That is, the bottom left corner corresponds to $-3.75 - 3.75i$, the top right corner corresponds to $3.75 + 3.75i$, etc.

Image 2, $w = z^2$

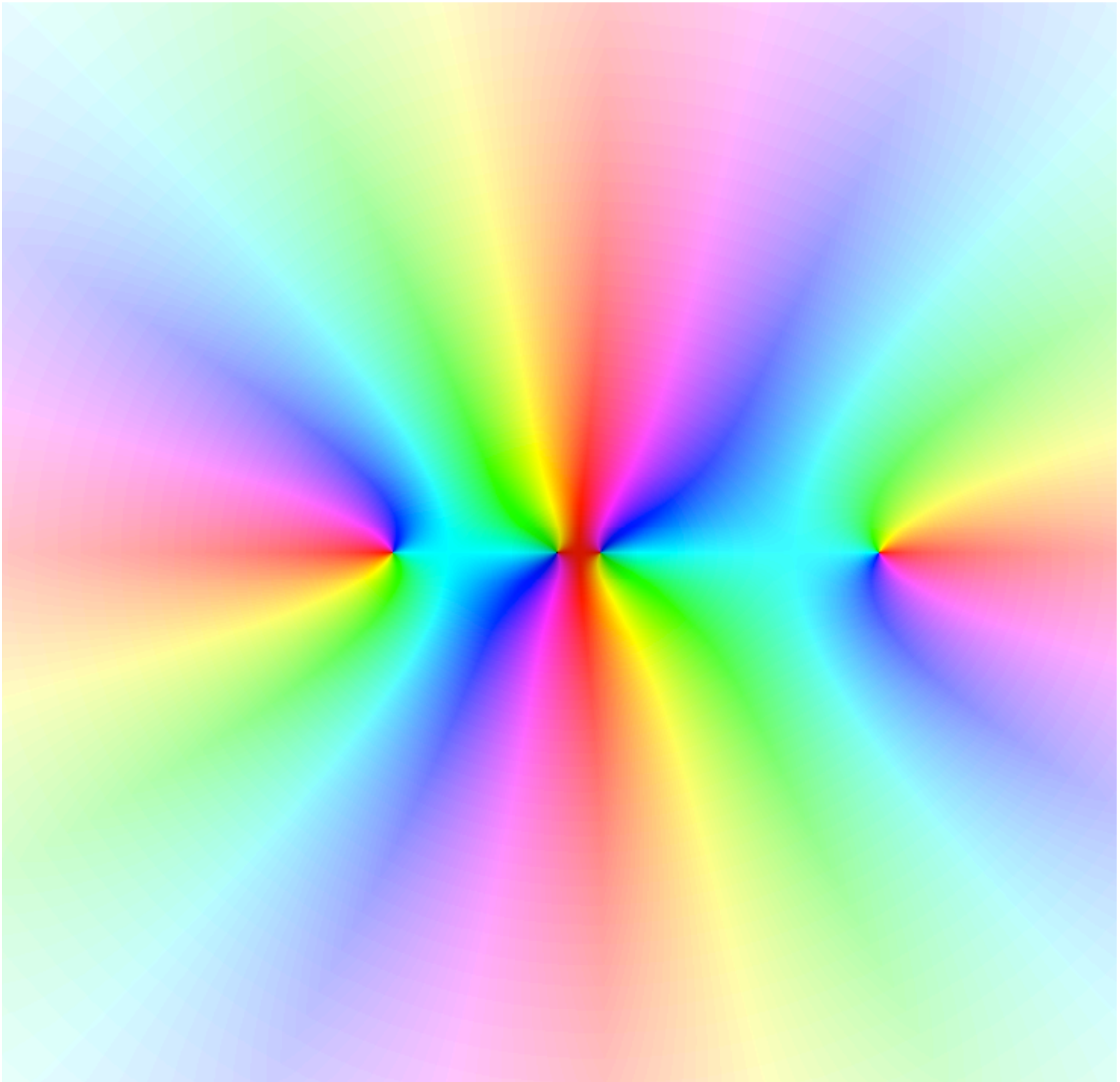


The plot range is $[-2, 2]$ for both components (real and imaginary)

Image 3, $w = z^5$

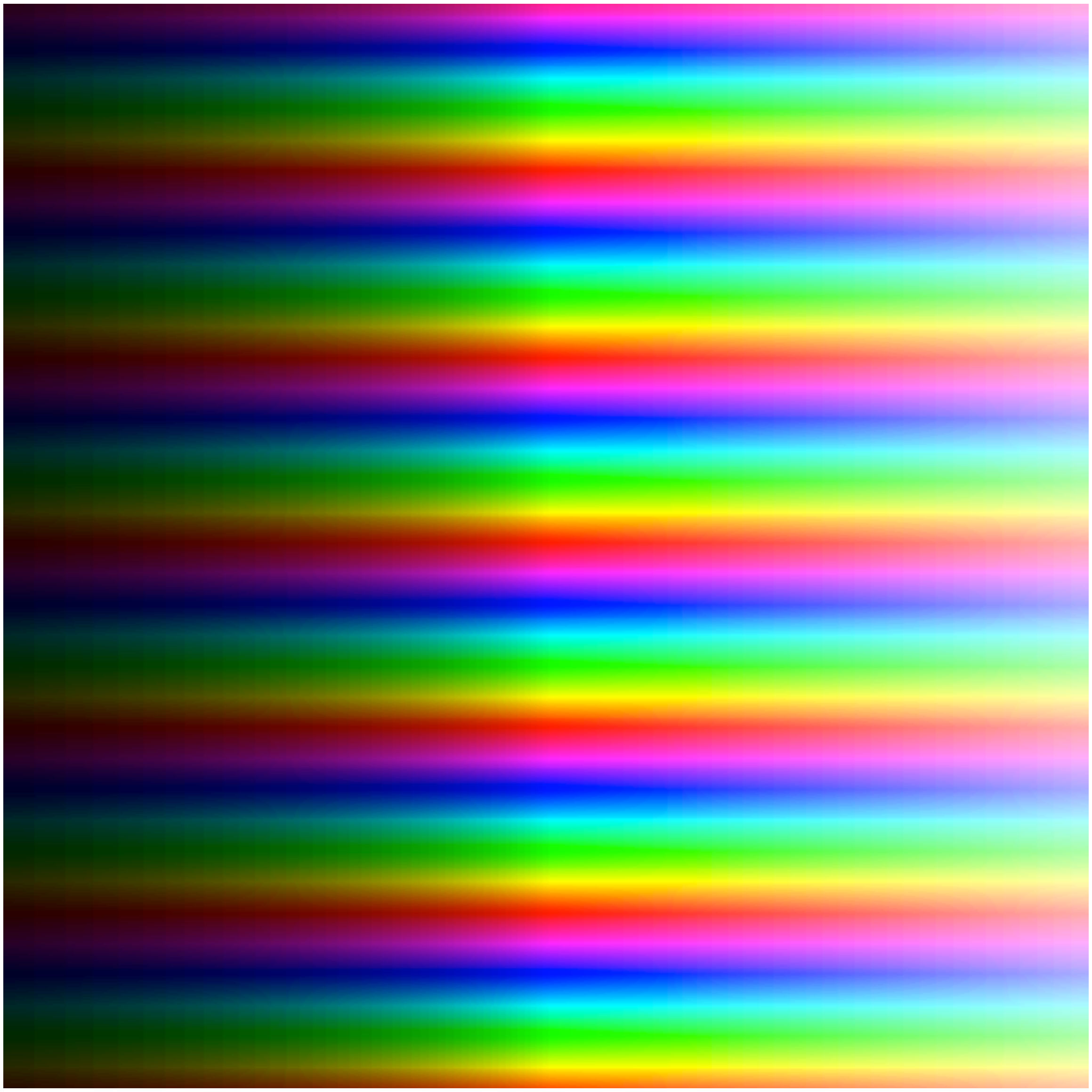
The interval is $[-10, 10]$ for both components. It is clear how larger numbers will form massive magnitudes to make the colors white.

Image 4, $w = 3z^4 - 4z^3 - 6z^2 + 2z$



This is a polynomial with real coefficients. Notice how every zero has every single color (complex argument) surround it. They are all simple zeroes, so each color touches them exactly once. The total number of zeroes, counted with multiplicity, is four. Notice how, far away from the central zeroes, each color is shooting out four times (red happens in four different directions), reflective of this fact. We will explore this later in the main work, when we talk about the argument principle.
The interval here is $[-3.5, 3.5]$ for both components.

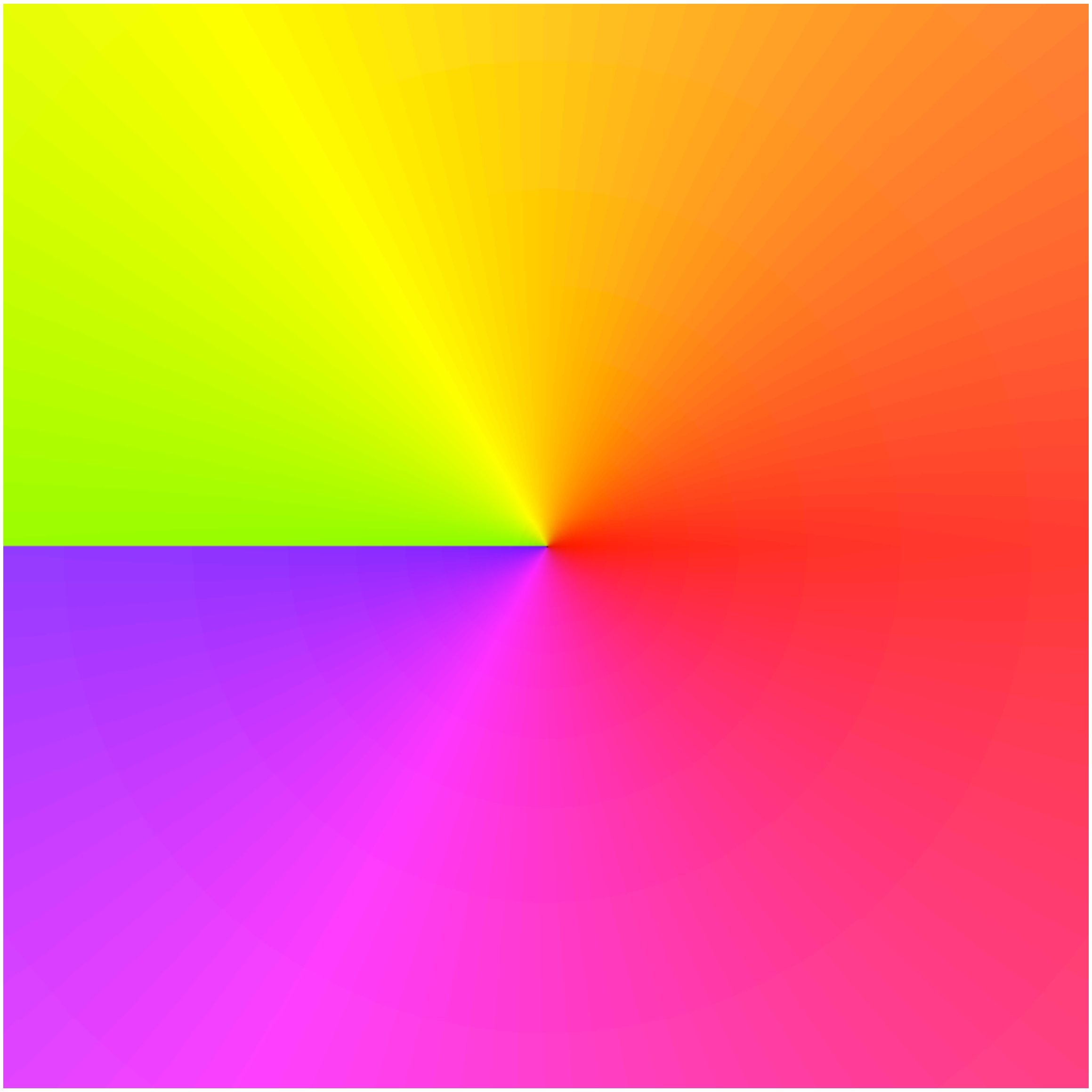
Image 5, $w = e^z$



The exponential function is clearly periodic when we go up $2\pi i$, hence the repeated "rainbow" of colors. It is indeed a rainbow because the complex exponential visits every complex argument equally as it cycles through the complex plane by changing the imaginary component of z. The shading is reflective of how the magnitude changes with the real component. Also notice that it never attains any zero value, but merely "gets close".

The interval here is $[-18, 18]$, and the brightness function has been rescaled to compete with the exponential function, so that it does not become white so quickly.

Image 6, $w = \sqrt{z}$

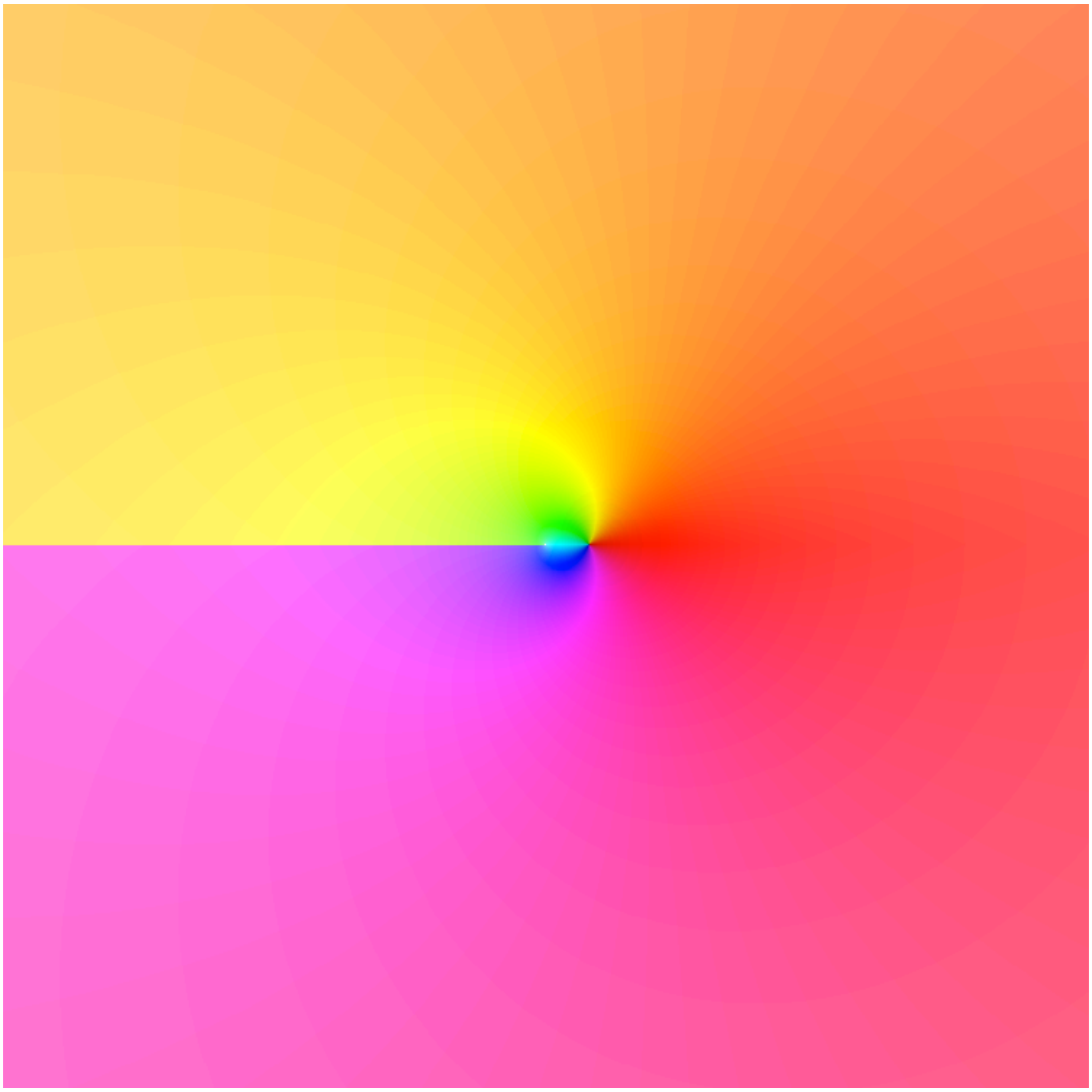


This is the first example of a function that suffers a branch cut along the negative real axis, because we had to choose which branch of the square root to take. The following two plots on the next two pages will also display this branch cut behavior. Notice, too, that the range is clearly restricted in the first two branch cut plots, as they will take no blue or cyan values, and the logarithm is also restricted (in its imaginary range), though this restriction is less obvious from the color plot.
The plot range is $[-18, 18]$ for both components.

Image 7, $w = z^{1/12}$

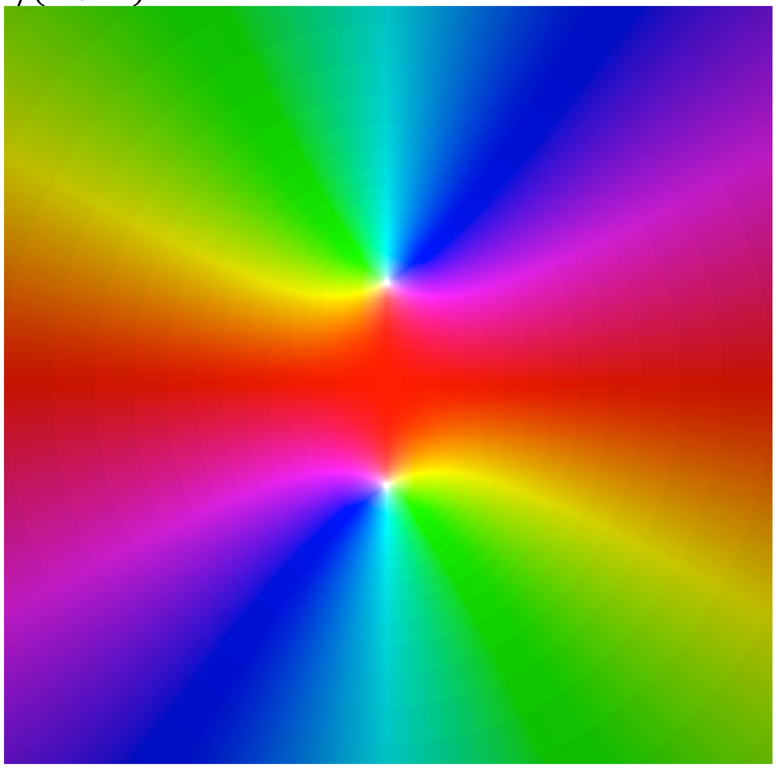
This image is NOT just a red block. It is the twelfth root, so we have to restrict our range to numbers with an argument between $-\pi/12$ and $\pi/12$, including $\pi/12$. Because of this, all of the numbers have arguments very close to zero, hence the almost uniform, red coloring. This restriction of the range makes the function to have a branch cut. Naturally, there is a (barely visible) branch cut along the negative real axis.
The plot range is $[-18, 18]$ for both components.

Image 8, $w = \ln(z)$



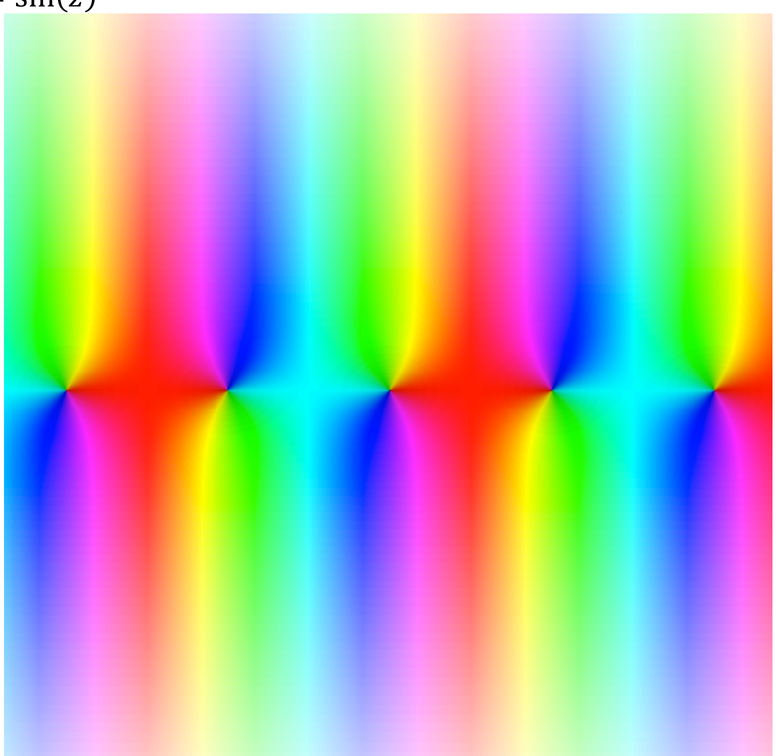
The logarithm boasts a simple zero at $z = 1$, and a discontinuity at $z = 0$. It is a bit hard to see that the range is restricted in the imaginary component to be between $-\pi$ and π. The pole, which is where the branch cut stems from, will not have one of every type of color touch it, unlike the simple zero, which will. The branch cut in the logarithm, like with the root functions, stems from the discontinuity of the argument function across the negative real axis.
The plot range is $[-12.5, 12.5]$ for both components.

Image 9, $w = 1/(1 + z^2)$



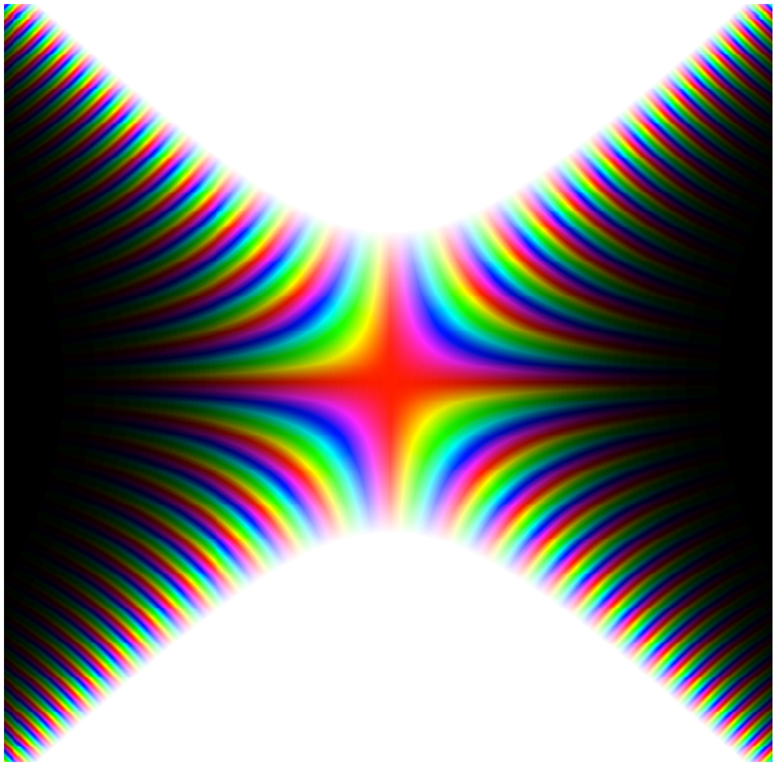
The plot range is $[-3.5, 3.5]$ for both components.

Image 10, $w = \sin(z)$



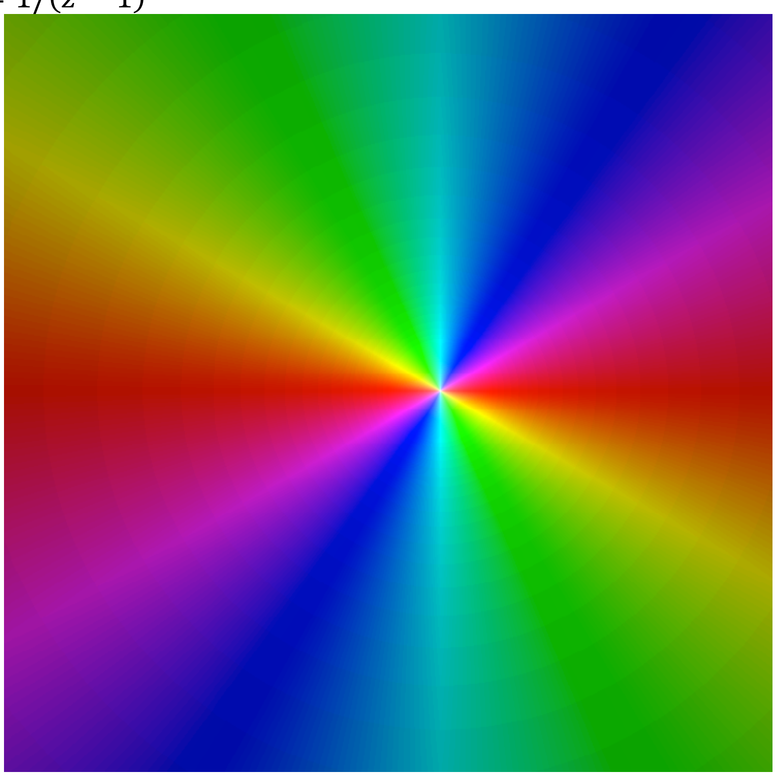
The plot range is $[-7.5, 7.5]$ for both components. Notice the horizontal periodicity.

Image 11, $w = e^{-z^2}$



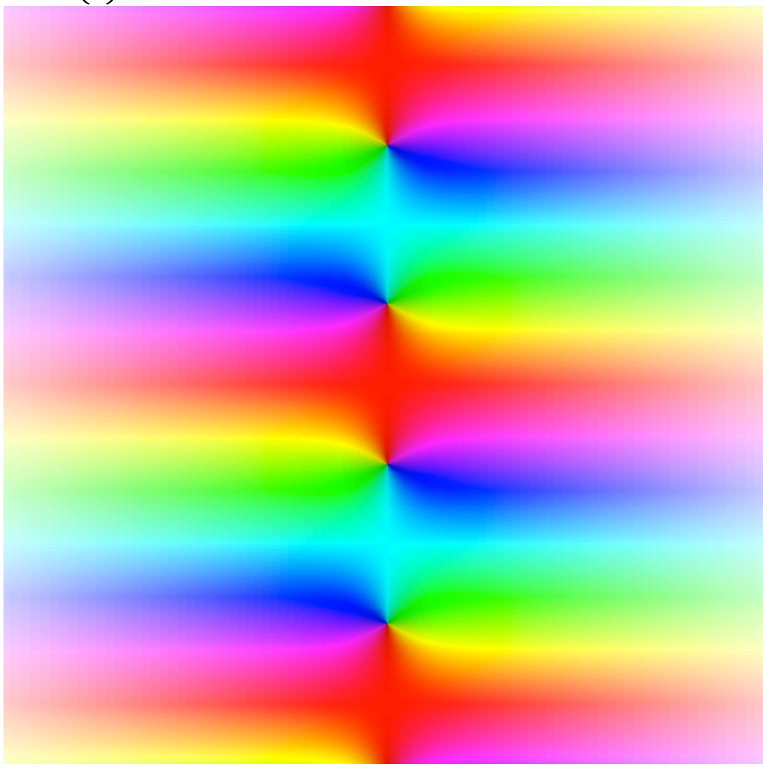
The plot range is $[-7.5, 7.5]^2$. This function is clearly *not* a bounded one.

Image 12, $w = 1/(z-1)^2$



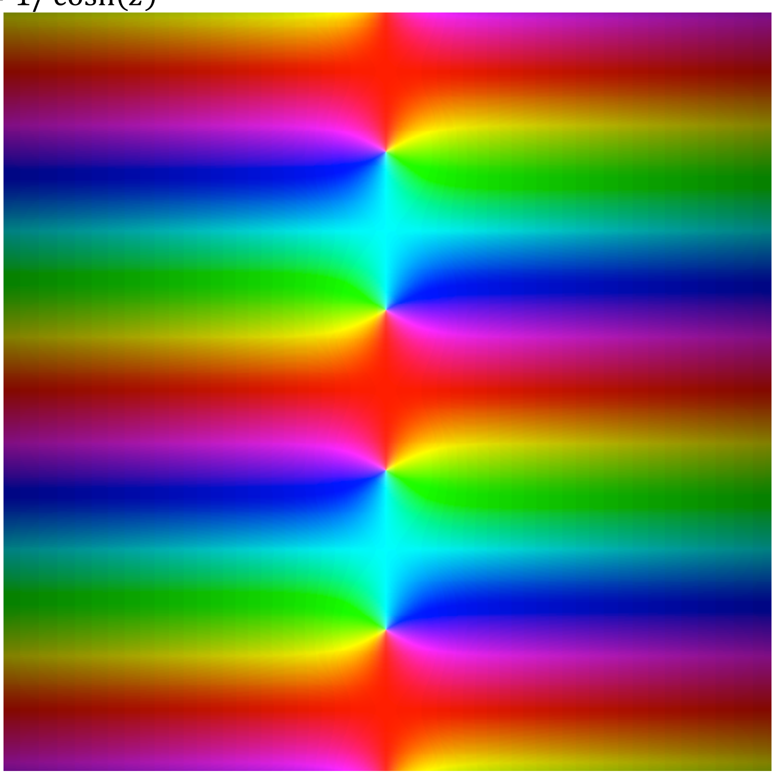
There is a double pole centered at $z = 1$. The plot range is $[-7.5, 7.5]^2$.

Image 13, $w = \cosh(z)$



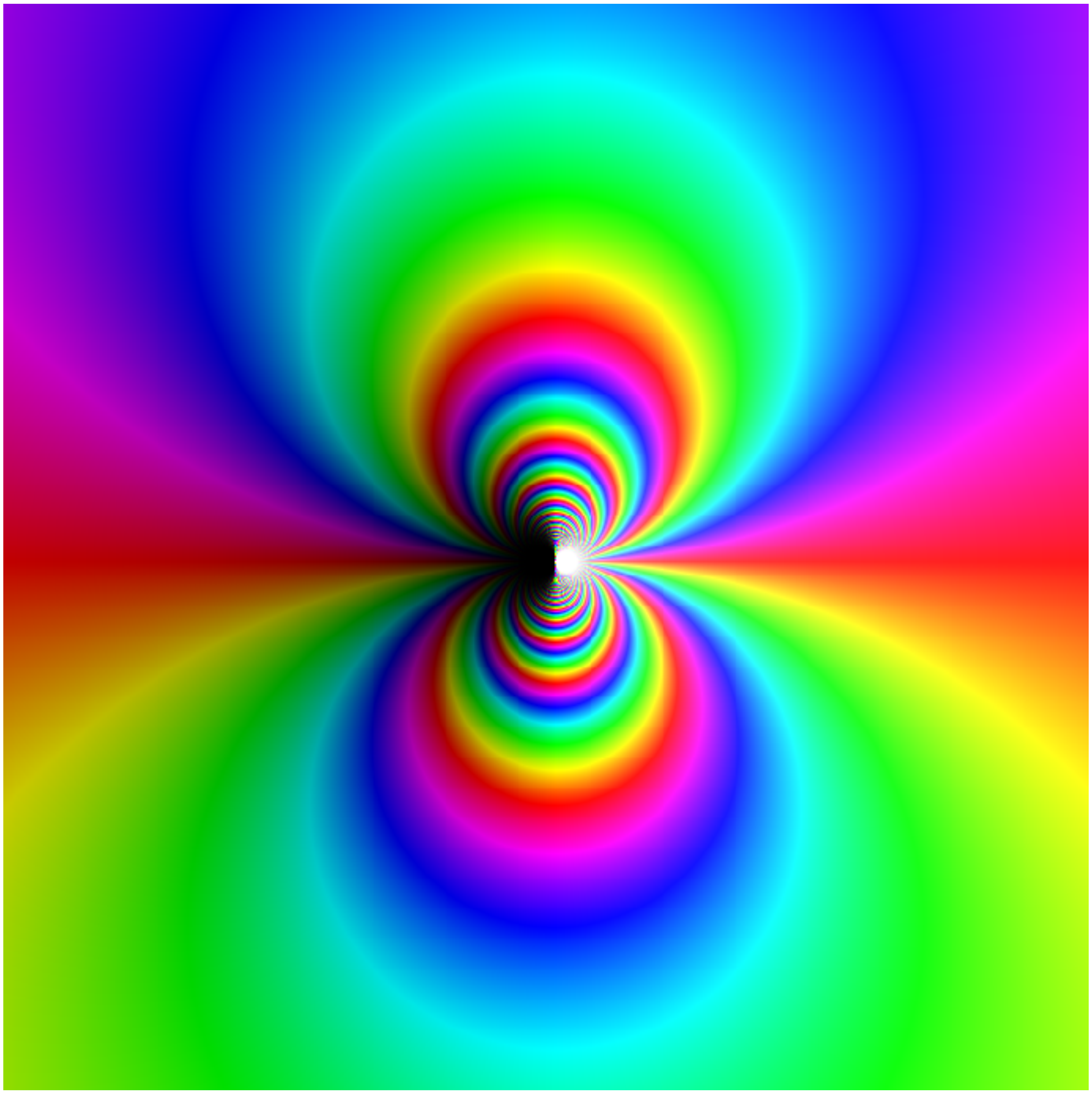
The hyperbolic cosine is a rotation of the cosine function's argument.

Image 14, $w = 1/\cosh(z)$



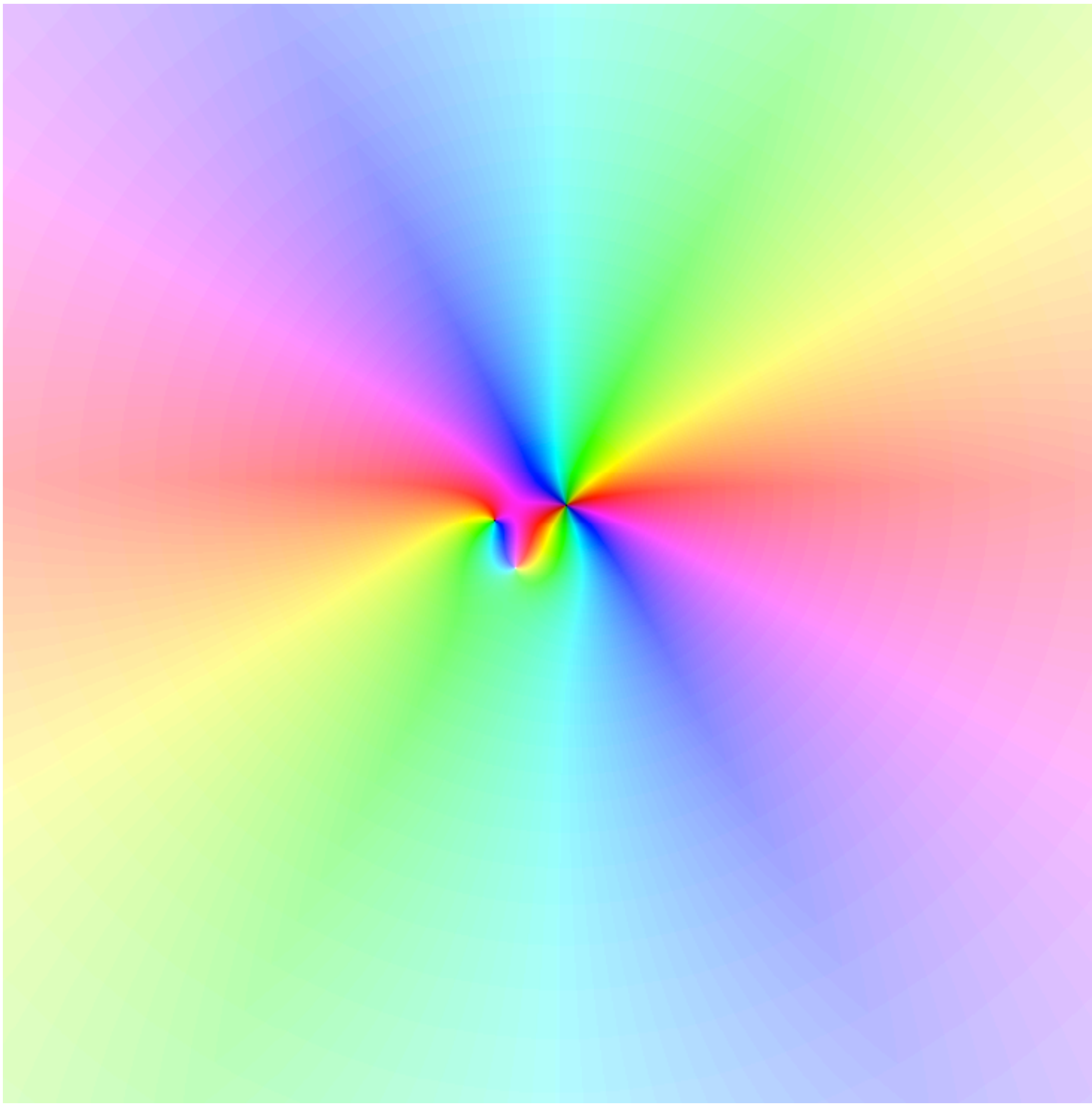
This is the reciprocal of the above. Both plot ranges are $[-7.5, 7.5]^2$.

Image 15, $w = e^{\frac{1}{z}}$



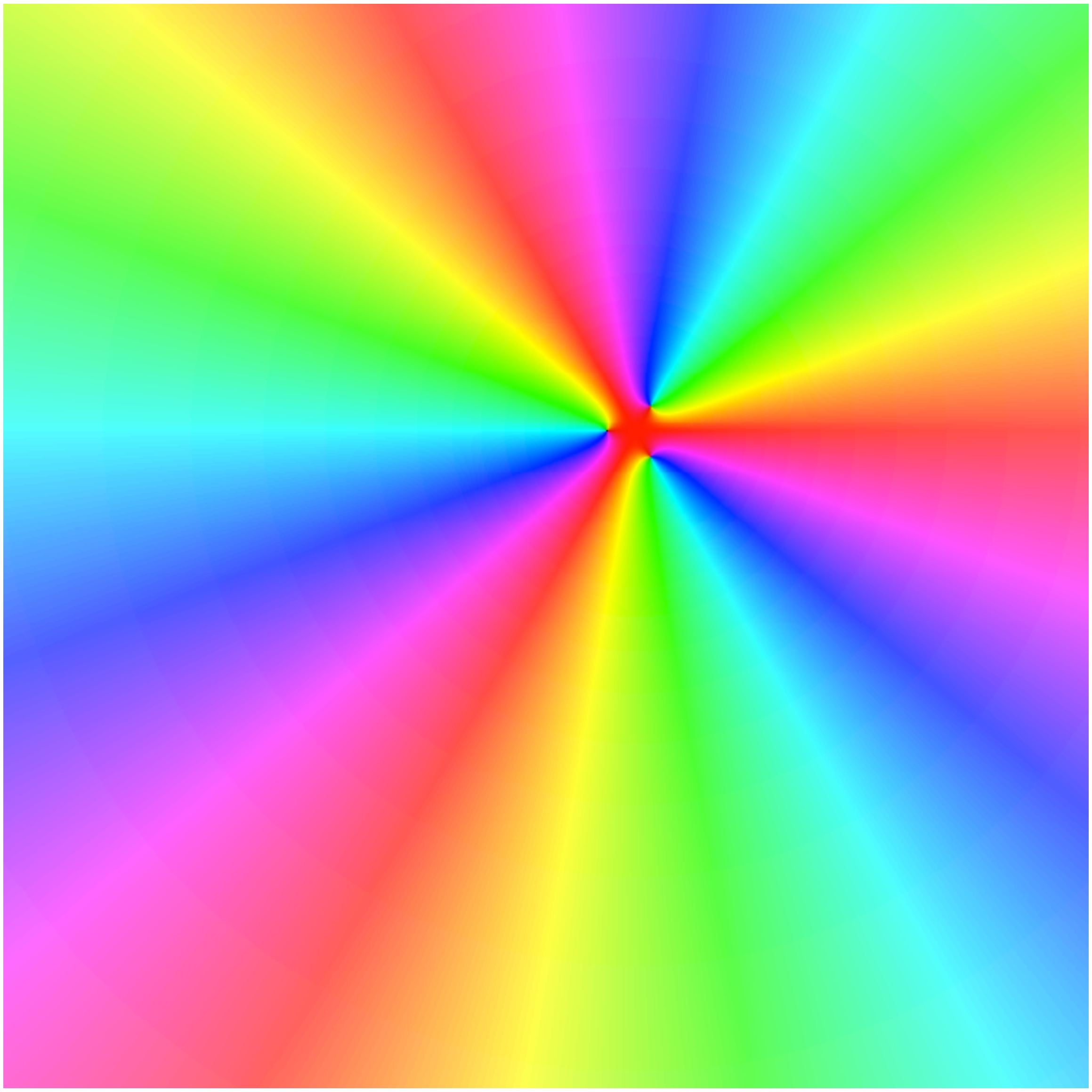
This serves as a classic example of an essential singularity at the origin. Notice how every single color appears very many (in fact, infinitely many) times as we approach the origin. Color plots can very elegantly show the power of the essential singularity in this way. It is very clear to see the role that "infinity" has here. The function begins to map more densely across the whole complex plane as we approach the origin.
The plot range is $[-.38, 38]$ for both components. Zooming out would show that one singularity, surrounded by a sea of well-behaved "reddish" color. All of the "chaos" is concentrated around the origin.

Image 16, $w = \big(z - (.4 + 1.5i)\big)^2 \big(z - (-2 + i)\big)\big(z - (-1.3 - .6i)\big)^{-1}$



This is an example of a rational function in the plane. There is a double root, as well as a simple root and a pole. The point where two of every color intersect is the double root, while the two points where every color intersects with every other are the simple root and the simple pole. There are thus three roots, counted with multiplicity, and one pole. This sums up to a winding number of two, so as we zoom out, we see two of every color emerge from the clump of roots and a pole. Counterclockwise around the root, the colors go RGB, and around the pole, they go RBG. This is always the case.
The plot range of this is $[-18, 18]^2$.

Image 17, $\left(z-(3+4i)\right)^3+1$



This image was only used to reiterate a proof that near any point z_0 where $f'(z_0)=0$, the function will take up certain values multiple times. Our function in question would be $\left(z-(3+4i)\right)^3$, and we are perturbing it by 1. It is clear that in some neighborhood of that center point, we have a root. By perturbing it less, the root will be closer to the center point.
The plot range of this is $[-37.5, 37.5]^2$.

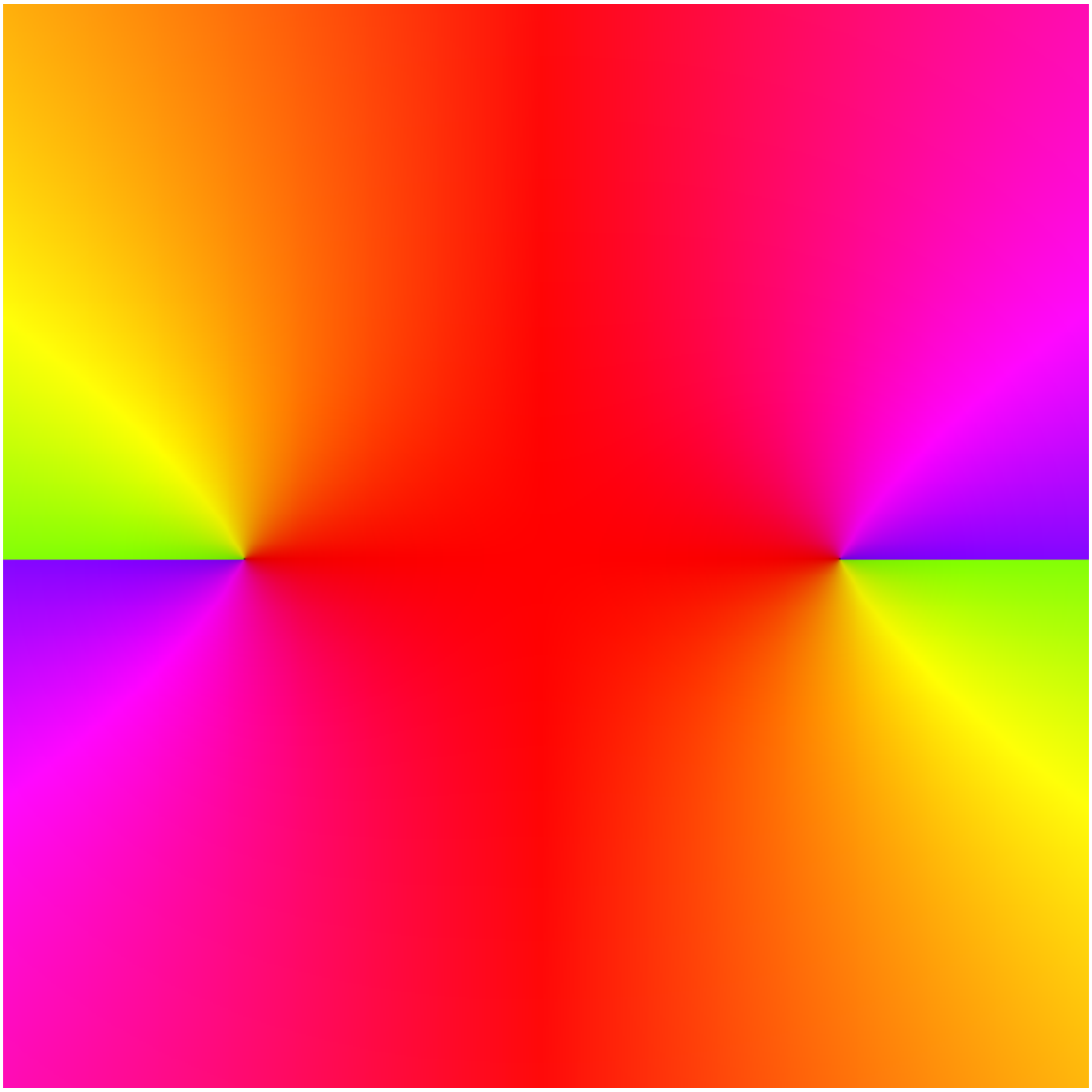
Image 18, $\Gamma(s)$

This is the classic image of the Gamma function, and one can see the rapid decay on one side, checked by the rapid growth on the other side. The poles can be seen on each of the non-positive integers.
The plot range of this is $[-18, 18]^2$.

Image 19, $\zeta(s)$

The Riemann zeta function is uniquely fascinating, and it is here that one can see its very modest behavior on the plane $\mathrm{Re}(s) > 1$. As well as this, the pole at $s = 1$ is visible, as are ten of the zeroes, all along the line $\mathrm{Re}(s) = 1/2$. The fact that we can see these ten zeroes will offer us no help in proving that they have a real part of one half.
The plot range of this is $[-37, 37]^2$.

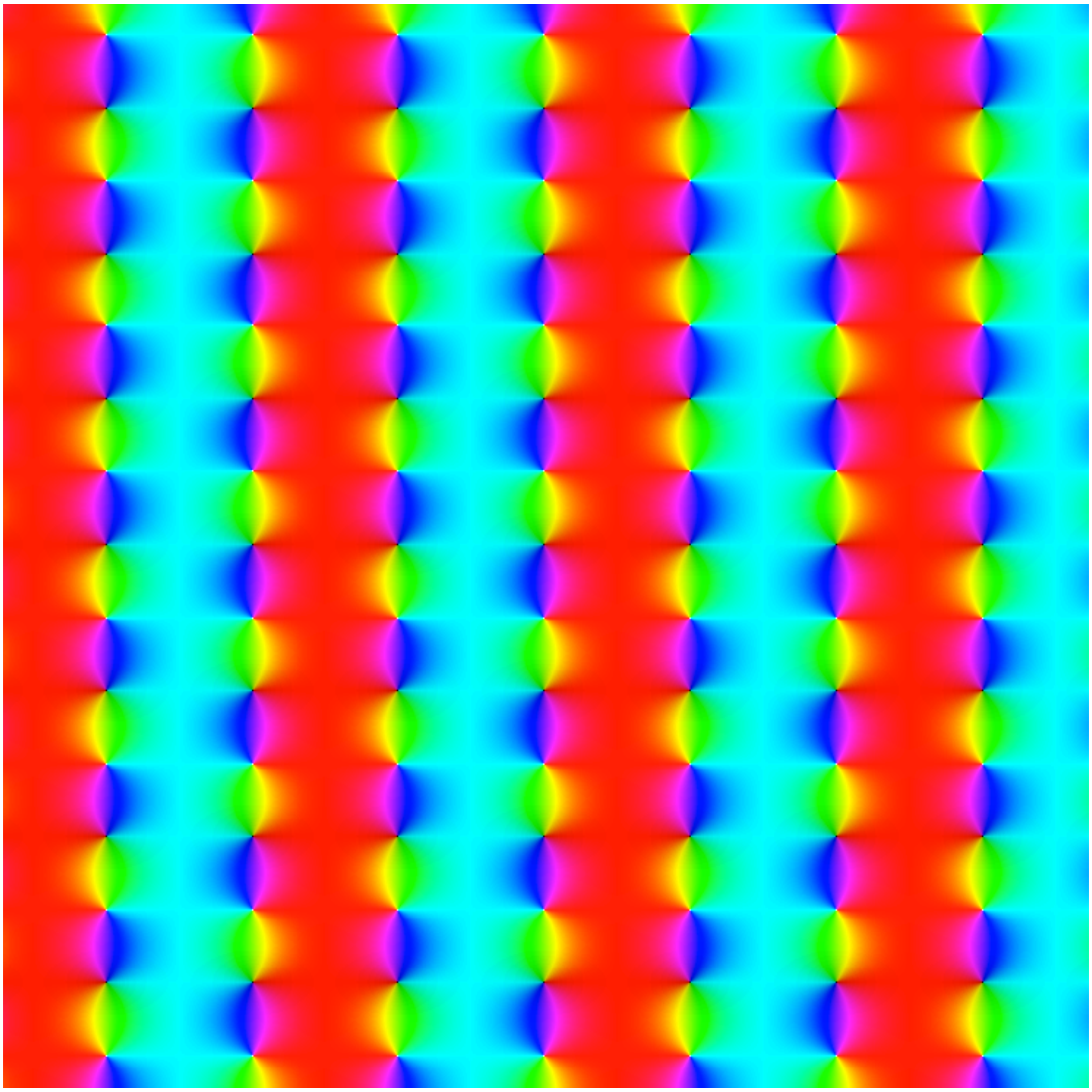
Image 20, $\sqrt{1-\zeta^2}$



This function, which suffers branch cuts, is one that we had to deal with when computing an elliptic integral. The graph at least illustrates the branch cuts, although it is not clear that the function makes the real axis with numbers greater than 1 and less than -1 become positive imaginary under its mapping. This fact is clear from the function definition.

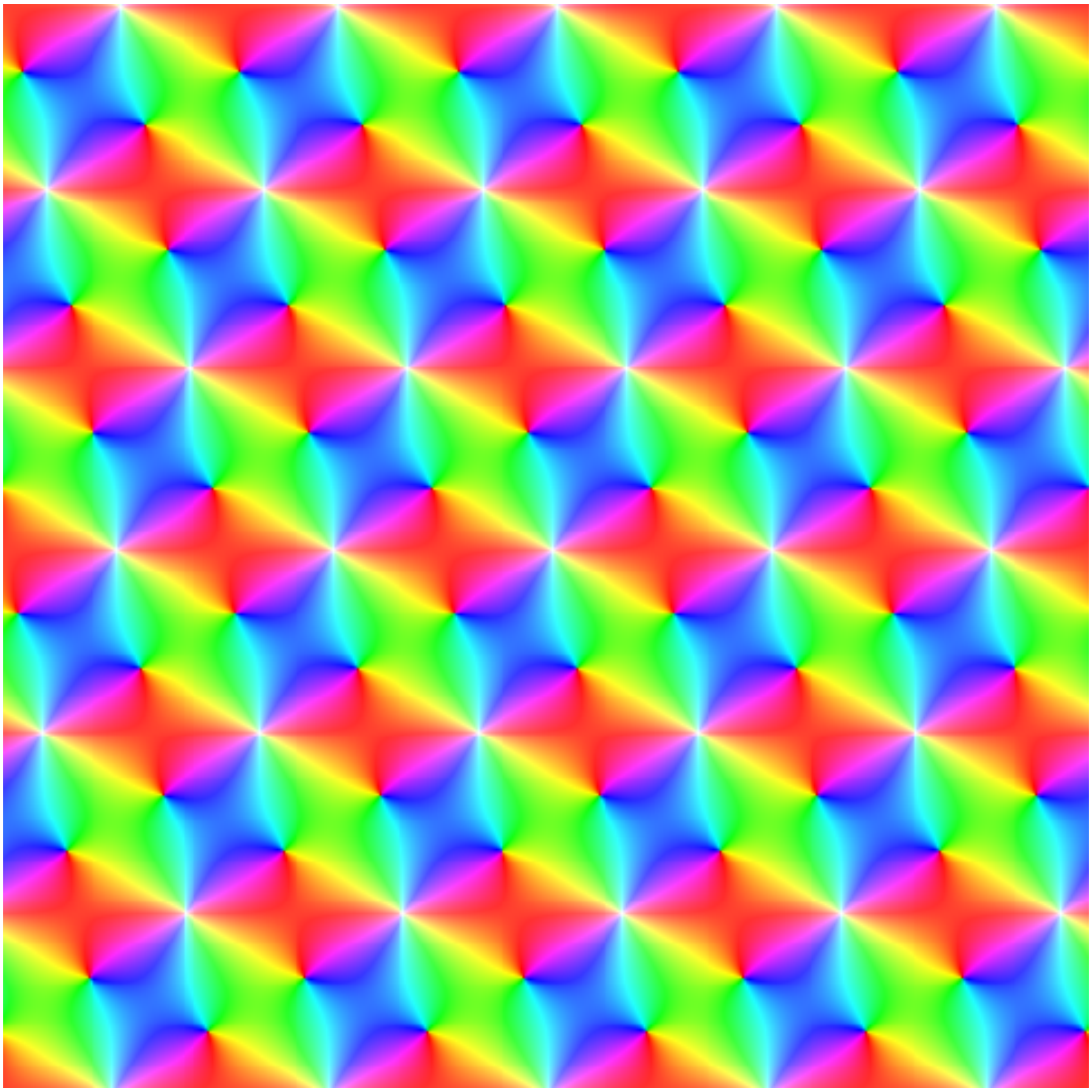
The plot range is $[-2, 2]$ for both components.

Image 21, $\mathrm{sn}(z)$



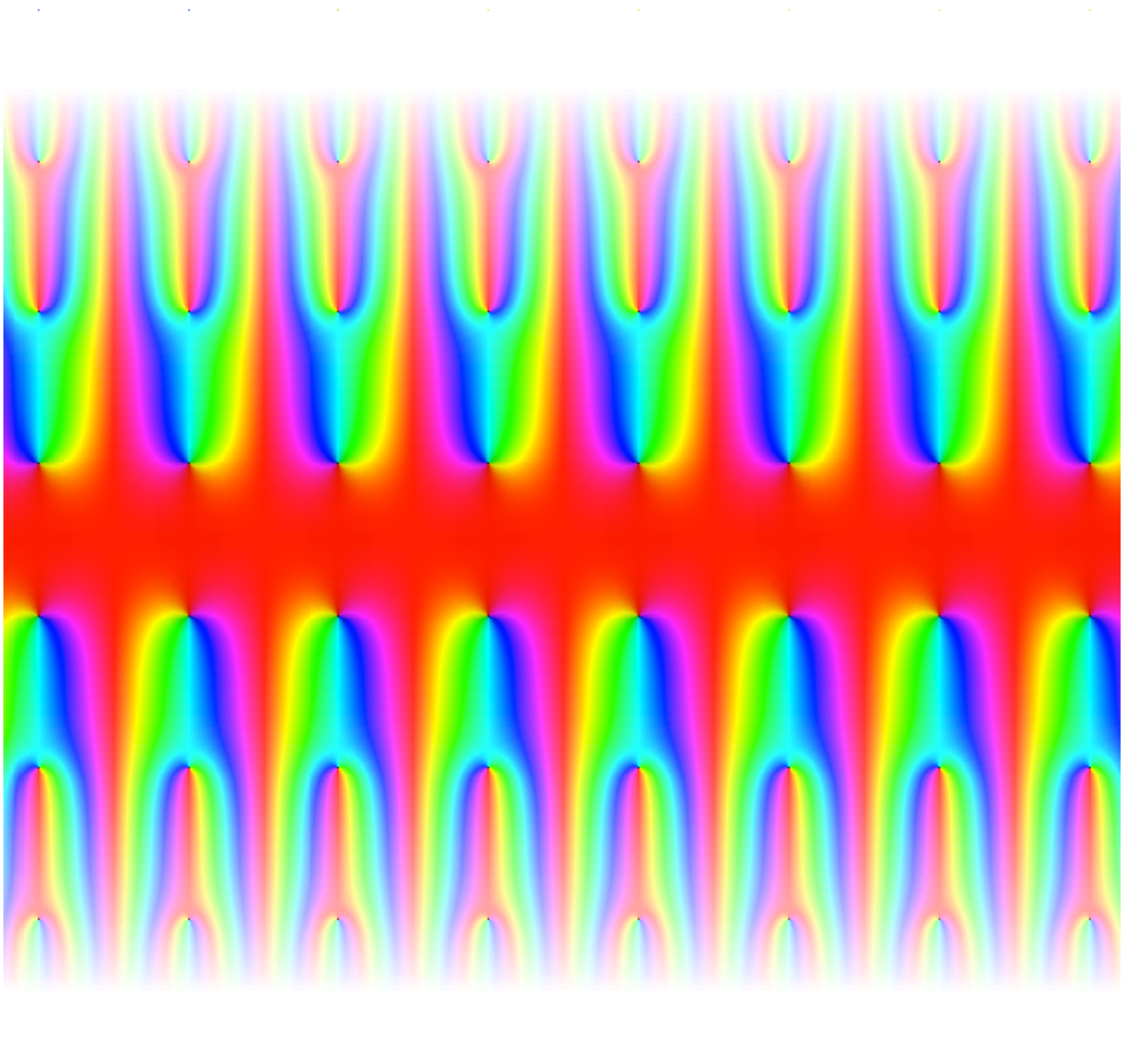
This is the elliptic function that arose as the inverse of the elliptic integral considered in the chapter. It is related to Jacobi's elliptic functions. The double-periodicity is clear. It is periodic with periods 1 and i in this case, although the sn can have different second periods besides i.
The plot range is $[-3.5, 3.5]^2$.

Image 22, $\wp(z, 0.34 + 0.87i)$



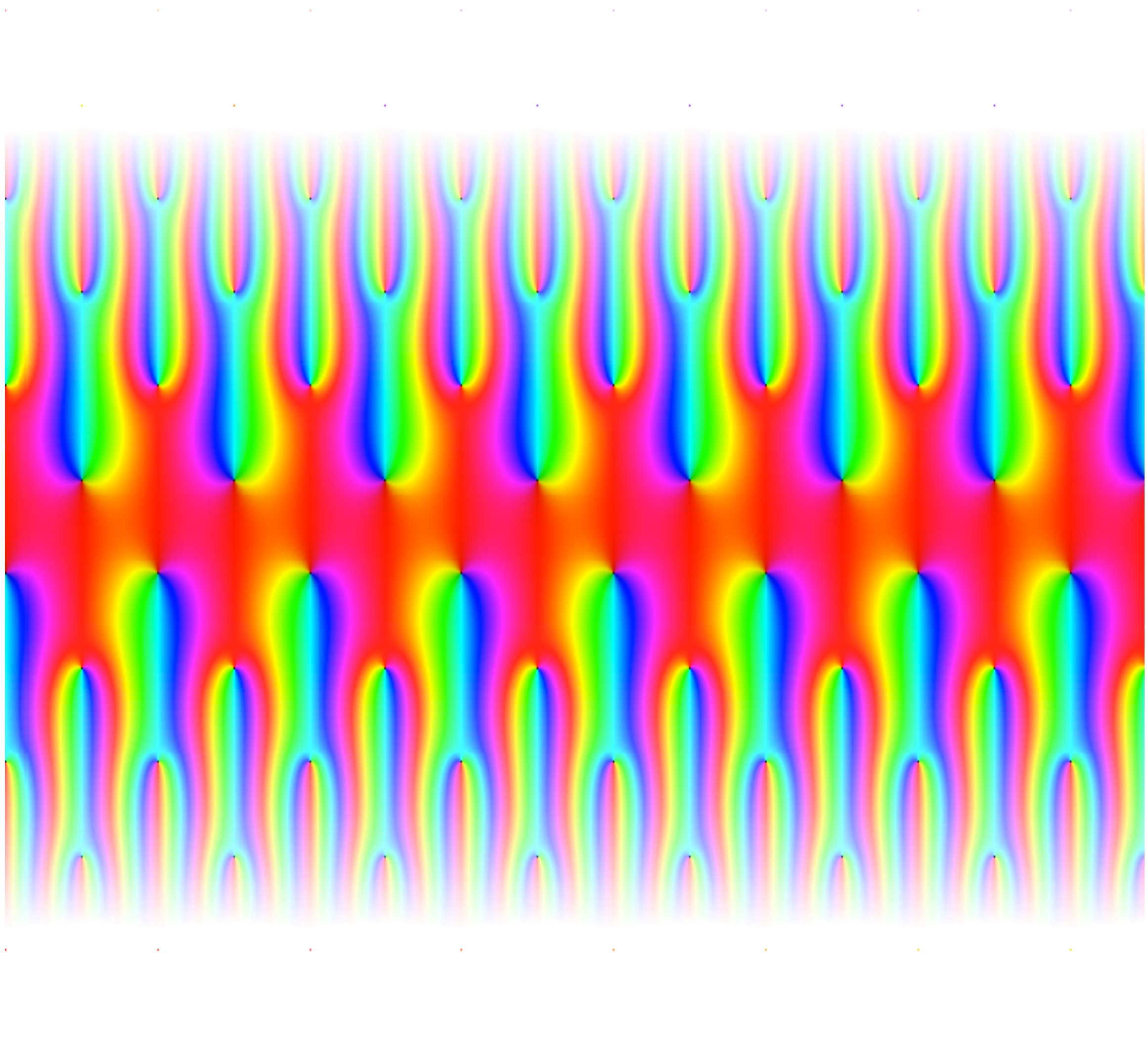
This is one example of the Weierstrass $\wp$ function. The lattice is clearly visible, and has periods 1 and $0.37 + 0.87i$. Changing the argument of τ will indeed change what the fundamental parallelogram is, and hence will change the tiling noticeably.
The plot range is $[-2.5, 2.5]^2$.

Image 23, $\Theta(z|i)$



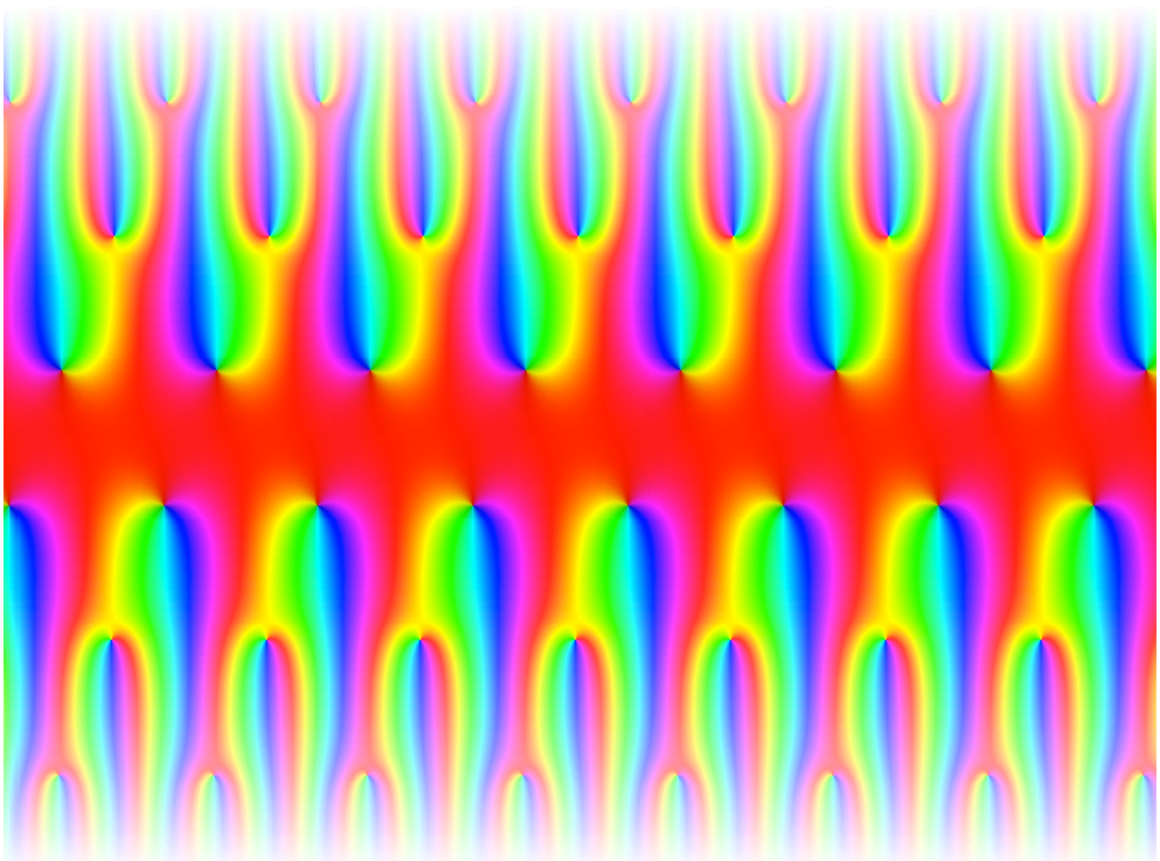
This is the Theta function when the second argument, τ, is set equal to i. The periodicity under $z \to z + 1$ is very clear.
Also, noticing how we have a series of "drips" as we go up vertically, so we can also *see* the quasi-periodicity under $z \to z + \tau$.
The plot range is $[-3.5, 3.5]^2$.

Image 24, $\Theta\ \left(z\middle|\frac{1}{2}+\frac{i}{2}\right)$



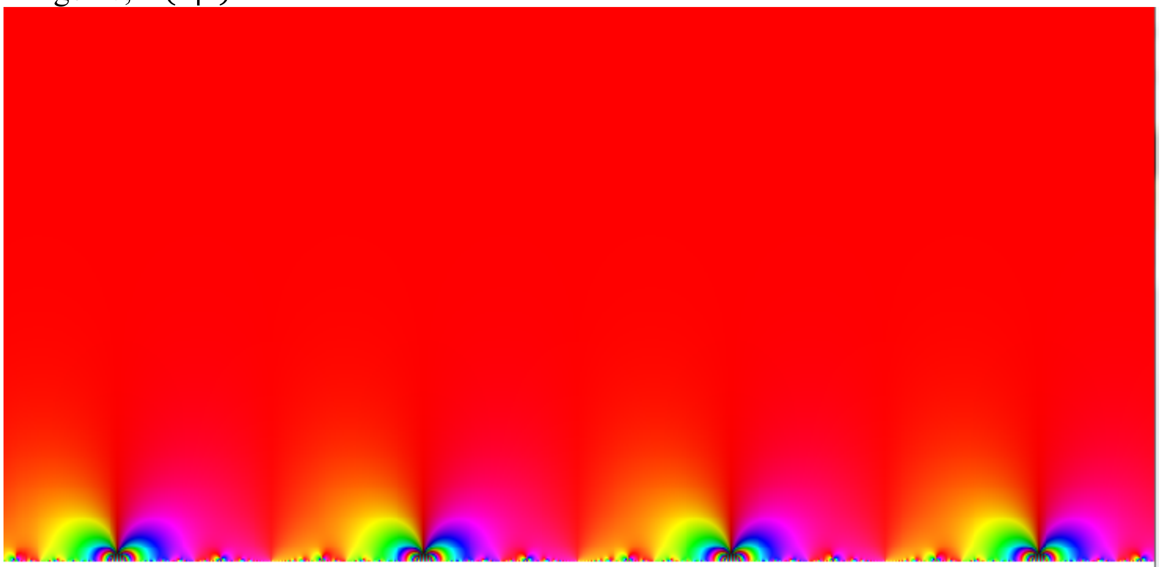
This is the Theta function for a different value of τ. Because τ has that half factor associated with it, it skews the way that the drips work.
The plot range is $[-3.5, 3.5]^2$.

Image 25, $\Theta(z|0.34 + 0.87i)$



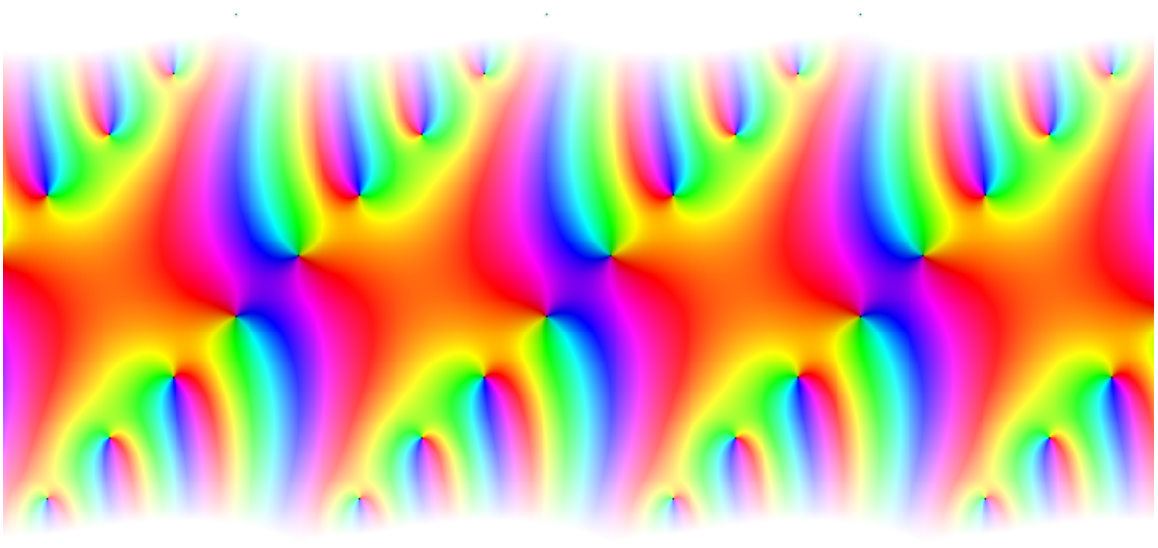
Again this is a Theta function with a different τ. In all of these, we can see the periodicity and quasi-periodicity.
The plot range is $[-3.5, 3.5]^2$.

Image 26, $\Theta(0|\tau)$



This image demonstrates the modular character of the Theta function. Notice the periodicity as we shift over horizontally. This clearly demonstrates the invariance under $\tau \to \tau + 2$. The transformation property under $\tau \to -1/\tau$ is *far* more difficult to see, but is going to be the reason for why there are “clumps” of color at the bottom.
The plot range is $[-3.5, 3.5]$ horizontally and $[0, 3.5]$ vertically.

Image 27, $\Theta(z|1.2 + 0.2\,\tau)$



Merely for the sake of beauty, choosing an argument of τ with small imaginary part in Theta will make it a flourish of colors.
The plot range is$[-2, 2]$ horizontally and $[-1.5, 1.5]$ vertically.