Galois Representations, Cuspidal Eigenforms, and Maass Forms at 1/4

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April 27, 2016

1 Introduction

This set of lecture notes will constitute both a general literature summary on the topic and an end-of-term paper for Spectral Geometry.

For particular modular forms g that are eigenfunctions of the Hecke operators (called cuspidal eigenforms of weight 1 and level N) we can relate the coefficients a_n in the Fourier series $g(\tau) = \sum_n a_n q^n$ to the characteristic polynomials of representations of certain Galois groups of finite extensions over $\mathbb{Q}$.

We will discuss Artin's Conjecture on the holomorphy of L functions. This has far reaching consequences about the correspondence between *odd* Galois representations giving cuspidal eigenforms of weight 1 and *even* representations giving Maass forms of eigenvalue 1/4 of weight 0.

2 Rings of Integers

Let $\mathbb{Q}$ be our ground field and consider a Galois extension $K/\mathbb{Q}$ of degree n. Let G be the corresponding Galois group. The ring of integers O_K of K is the integral closure of $\mathbb{Z}$ in K (the algebraic integers in K). This is a Dedekind domain, and so all nontrivial prime ideals are maximal. Moreover, every ideal can be factored into a product of prime ideals.

Example 1. *In $\mathbb{Z}[i]$, we have $(2) = (1+i)(1-i)$. Note how although 2 was prime in $\mathbb{Z}$, it is no longer prime in our ring of integers $O_K = \mathbb{Z}[i]$.*

So which rational primes have repeated factors when they split in O_K?

$$(p) = \pi_1^{e_1} \dots \pi_r^{e_r}$$

Then we say p is ramified in K if $\exists i : e_i > 1$.

Proposition 1. *The ramified primes are precisely those that divide the* relative discriminant $(\det(\sigma_i b_j))^2$ *where* σ_i, b_j *are the Galois automorphisms of* K *and* b_j *are the elements of the integral basis for* K*.*

Corollary 1. *There are only finitely many ramified primes.*

Let's say p is unramified. If $\pi | p$ then all $\sigma_i \pi | p$. In fact G acts transitively on the primes π dividing p. For a given prime ideal π, the set of all $\sigma \in G$ that stabilize this ideal is called the decomposition group $G_\pi = \text{Stab}_G(\pi)$. By transitivity, for all the $\pi | p$ these groups are all conjugate in G. We see from definition that G_π descends to act on the quotient field O_K/π.

Since $(p) \subseteq (\pi)$, we get that (p) is in the kernel of the quotient. We therefore have that the integers map to either 0 or $\mathbb{Z}/p\mathbb{Z}$ and since $(\pi) \neq O_K$, the integers map to $\mathbb{Z}/p\mathbb{Z}$. O_K/π is therefore some finite extension (say of degree f) over $\mathbb{F}_p$. Simply by counting dimension, the degree of the extension $(O_K/\beta)/(\mathbb{Z}/p\mathbb{Z})$ is $f = n/r$.

The Galois group of this finite field extension is cyclic of order f.

The automorphisms are generated by some element $x \mapsto x^p$, and we denote this corresponding element in G_π by Frob_π. So $\text{Frob}_\pi(\alpha) = \alpha^p \mod \pi$.

Proposition 2. *The elements* Frob_π *are conjugate in* G *for* $\pi | p$*. We can thus define* Frob_p *as the conjugacy class of* Frob_π *for* $\pi | p$*.*

Proposition 3. *For* p *unramified,* $\text{Frob}_p = 1$ *iff* p *is split in* K*. If* K *is the splitting field of* f*, this corresponds to* f *splitting completely when reduced mod* p*.*

I didn't clarify what it means for unramified p ***to split completely in lecture:*** *A prime splits completely if* $(p) = \pi_1 \dots \pi_n$ *and* $O_K/\pi_i = \mathbb{Z}/p\mathbb{Z}$ *for each* i*. This is consistent with* $\text{Frob}_p = 1$ *as there are no nontrivial automorphisms on this field. Otherwise, we'd clearly have nontrivial elements in* Frob_π *for some* π*, and therefore* Frob_p *is nontrivial.*

Example 2. *In* $\mathbb{Z}[i]$*, the rational prime* 3 *remains prime while* $2 = (1+i)(1-i)$ *is split. The Galois group of* $\mathbb{Q}(i)/\mathbb{Q}$ *is* $\mathbb{Z}_2$*, and we have* $\text{Frob}_3 = \langle x \rangle$ *while* $\text{Frob}_2 = 1$*. This can be checked. For* $p = 3$*:*

$$(a + bi)^3 \equiv a^3 - ib^3 \equiv a - ib \equiv \overline{a + ib} \mod 3 \quad (1)$$

and $a, b \in \mathbb{Z}$. *This means* $\text{Frob}_3 = -1$. *On the other hand we can check:*

$$(a + bi)^5 \equiv a^5 + b^5 i = a + bi \text{ mod } (2 + i) \tag{2}$$

for any a, b, *so* $\text{Frob}_5 = 1$

Example 3. *Consider* $\mathbb{Q}(i, \sqrt[4]{2})/\mathbb{Q}$, *this has Galois group* D_8 *and thus admits a two-dimension representation into* $\text{GL}_2(\mathbb{C})$.

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now for the unramified rational primes, we have a characterization of conjugacy classes:

$$\text{Frob}_p \textit{is the conjugacy class of} \begin{cases} 1 & \textit{if } p = a^2 + 64b^2 \\ r^2 & \textit{if } p = a^2 + 16b^2, b \textit{ odd} \\ rs & \textit{if } p \equiv 3 \textit{ mod } 8 \\ r & \textit{if } p \equiv 5 \textit{ mod } 8 \\ s & \textit{if } p \equiv 7 \textit{ mod } 8 \end{cases}$$

The trace of the representation is nonzero for $1, r^2$ *where it is* 2 *and* -2 *respectively.*

We will connect this to a modular form in section 4.

3 Modularity for GL_1 and L functions

If $K/\mathbb{Q}$ is a quadratic extension, we can write $K = Q(\sqrt{d})$ for some d. Whether d is positive or negative will make a difference in the overall treatment. We know that the Legendre symbol may be viewed as a one-dimensional character into $\text{GL}_1(\mathbb{C})$ from which we can form the associated Dirichlet Zeta (or Hecke L-function):

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}}$$

Artin generalized these one-dimensional representations to GL_2, and defined the Artin L-function for more general representations:

$$L(s, \rho) = \prod_p \frac{1}{\det(I - p^{-s}\rho(\text{Frob}_p))}$$

The representations for Galois groups of quadratic fields into $\mathbb{C}^\times$ can be induced to give the two-dimensional dihedral representations discussed in the next section. Hecke's use of a quotient of $\mathbb{Z}[i]$ to reproduce the coefficients given from the Galois representation for D_8 play on these ideas of "inducing from C_2 to D_{2n}". This is the reason for why the Dihedral representations are best understood among the finite subgroups of $\mathrm{GL}_2(\mathbb{C})$.

As for the Artin L-function, we have the following conjecture:

Conjecture 1 (Artin)**.** *For a nontrivial representation ρ, $L(s, \rho)$ is holomorphic on the whole complex plane.*

This is known for when the representation is 1-dimensional, for the Hecke/ Dirichlet L-functions. It is also known for representations *induced* from one dimensional representations (so dihedral).

Why does this matter? The holomorphy of $L(s, \rho)$ combined with certain other conditions will guarantee that the corresponding modular form $\sum_n a_n q^n$ has the right transformation properties and is an eigenform.

Worth Noting: Brauer's theorem on induced characters (from representation theory, c.f. Serre's "Linear Representations of Finite Groups") shows that any Artin L-function can be expressed as a ratio of Hecke L-functions, so any $L(s, \rho)$ is automatically meromorphic in the complex plane for nontrivial ρ.

4 Modular Forms and Galois Representations

Our familar theta function satisfies:

$$\theta(2\tau)^2 = \sum_{a,b \in \mathbb{Z}} q^{a^2+b^2} \tag{3}$$

We'll now see a specific example of a more general construction of modular forms involving the rings of integers in quadratic extensions. For a given quadratic extension's ring of integers, say $\mathbb{Z}[i]$, pick a nonzero element $\alpha \in \mathbb{Z}[i]$. On the quotient $\mathbb{Z}[i]/(\alpha)$ define the character:

$$\chi : (\mathbb{Z}[i]/(\alpha))^\times \to \mathbb{C}^\times \tag{4}$$

Then extend χ to $\mathbb{Z}[i]$ by letting it be 0 for elements sharing a factor with α. In this way, we can extend χ to a multiplicative function on $\mathbb{Z}[i]$.

Proposition 4 (Hecke)**.** *Under the above construction*

$$\theta_\chi(\tau) := \frac{1}{4}\sum_{a,b\in\mathbb{Z}} \chi(a+bi)q^{a^2+b^2} \tag{5}$$

is a modular form of weight 1 *and level* $4|\alpha|^2$*. If* χ *is not trivial, then we get a cusp form.*

Note $\chi(i)$ must have order dividing 4. In fact, since $(i) = \mathbb{Z}[i]$, so we can act by the four powers of i on $(a+bi)$ to produce 4 different elements all with the same value for a^2+b^2. If $\chi(i) \neq 1$ we'd get cancellation in the sum. Therefore it is in fact necessary that $\chi(i) = 1$.

Example 4. *Let* $\alpha = 8$ *so then the finite abelian group* $\mathbb{Z}[i]/(8\mathbb{Z}[i])^\times$ *has generators* $3, 5, i, 1+2i$ *(orders* $2, 2, 4, 4$*). Let* χ *map the first three to* 1 *and the last one to* i*. Then* θ_χ *is a modular form of level* 256*. By direct calculation and casework, we get that the pth Fourier coefficients of this modular form are precisely*

$$a_p = \mathrm{Tr}(\rho(\mathrm{Frob}_p)) \tag{6}$$

Where ρ *was the dihedral representation from example 3. This is a hint at a deep relationship between Galois represenatioons and modular forms.*

More generally, we can work towards making a broader statement between Hecke eigenforms . Firstly, if we have g is an eigenform (of the Hecke operators) of weight k and level N and there is a Dirichlet Character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ so that we get the following "twisted" expression:

$$g(\gamma\tau) = \chi(d)(c\tau+d)^k g(\tau), \forall\gamma \in \Gamma_0(N) \tag{7}$$

We say χ is the character of g (or *nebentypus* if you want to be *really* fancy).

Proposition 5. *If* $g(\tau)$ *above is a Hecke eigenform then we can write an associated Artin L-function from the coefficients.*

$$L(g,s) = \prod_{p\ \mathit{unramified}} \frac{1}{1 - a_p p^{-s} + \chi(p)p^{k-2s}} \prod_{p\ \mathit{ramified}} \cdots \tag{8}$$

This corresponds to a type of Mellin transform between L *and* g*.*

We want to see if this is in fact an Artin L-function for some representation of a finite Galois group.

So from the other side, let $G = \mathrm{Gal}(K/\mathbb{Q})$ admit a representation $\rho : G \to \mathrm{GL}_2(\mathbb{C})$. Since the characteristic polynomial $p(t)$ is a class function, we get

$$\det(I - \rho(\mathrm{Frob}_p)p^{-s}) \tag{9}$$

Is the characteristic polynomial of the conjugacy class Frob_p evaluated at $t = p^{-s}$. In a manner similar to the product form of the Riemann zeta:

$$L'(s, \rho) = \prod_p \frac{1}{\det(I - \rho(\mathrm{Frob}_p)p^{-s})} := \sum_n \frac{\lambda_\rho(n)}{n^s} \tag{10}$$

where special care is taken for the finite few p that are ramified.

Now we get to the critical result by Deligne and Serre (written as in Weinstein), showing that every Hecke eigenform comes from a $2D$ (odd) Galois representation.

(To make this clear, from each appropriate eigenform, we can get a Galois representation, *not* the other way around. A form of a converse to this shall follow the theorem.)

Theorem 1 (Deligne & Serre)**.** *Let $g(\tau)$ be as above: weight $k = 1$, level N, with corresponding character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.*

Then there exists an odd, irreducible Galois representation of some finite extension $K/\mathbb{Q}$ so that for every ℓ coprime to N,

$$\rho_g : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

is unramified at ℓ and the characteristic polynomial of $\rho_g(\mathrm{Frob}_p)$ can be written $x^2 - a_p(g)x + \chi(p)$.

(There are some details about the finite points of ramificaiton)

For higher weight Hecke eigenforms forms, we can also form such representations, but they are now p-adic.

Theorem 2 (Deligne & Serre, p-adic)**.** *For a number field F and a prime $\mathfrak{p}$ of F, let $F_\mathfrak{p}$ dene completion with respect to the $\mathfrak{p}$-adic metric.*

Let $g(\tau)$ be as above: weight $k \geq 2$, level N, with corresponding character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

For all primes $\mathfrak{p}$ in F, there exists an odd, irreducible Galois representation of some finite extension $K/\mathbb{Q}$ so that for every ℓ coprime to N and $\mathfrak{p}$,

$$\rho_{g,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\mathfrak{p})$$

is unramified at ℓ and the characteristic polynomial of $\rho_g(\mathrm{Frob}_p)$ can be written $x^2 - a_p(g)x + \chi(p)$.

This second theorem can be used to associate p-adic representations to eigenforms such as the modular discriminant Δ.

4.1 The Case of A_5: An Icosahedral Eigenform

Although A_5 does not sit inside $\mathrm{GL}_2(\mathbb{C})$, an extension of it by C_4, $\tilde{A}_5$, does. We know for a weight one eigenform we have some odd irreducible Galois (Artin) representation (and therefore some finite subgroup of $\mathrm{GL}_2(\mathbb{C})$) corresponding to it. In the other direction, do we have a form for $\tilde{A}_5$?. In fact, yes [Buhler]. Take

$$p(x) = x^5 + 10x^3 - 10x^2 + 35x - 18.$$

This has Galois group A_5. We can make a *projective* representation of A_5 in $\mathrm{PGL}_2(\mathbb{C})$. From this, we then make an appropriate extension of A_5 so that the non-projective representation in $\mathrm{GL}_2(\mathbb{C})$ has image isomorphic to $\tilde{A}_5$. That is, we let K, the splitting field of p be the field fixed by the kernel of the representation to PGL_2, and let an additional extension of K be permuted by the action of C_4 to form $\tilde{A}_5$ as the total Galois group.

From studying Frob_p for different p and using the fact that a_n are multiplicative, we obtain a list of coefficients: a_ℓ so that

$$\sum_\ell a_\ell q^\ell = q - iq^3 - i\phi q^7 - q^9 + \phi q^{13} + i(1-\phi)q^{19} - \phi q^{21} + \cdots = g(q)$$

The multiplicative structure shows itself in the coefficients above.

It can be shown [Buhler] that this is a modular eigenform of weight 1 and level $N = 800$. It was a difficult thing to prove that such a set of coefficients arising from an icosahedral representation has modular character.

Worth noting: an unramified prime splits iff $\mathrm{Frob}_p = 1$ iff the discriminant of the characteristic polynomial of $\rho\mathrm{Frob}_p$ is zero iff $a_p^2 = 4\chi(p)$.

So Serre and Deligne's theorem proves that given a cuspidal eigenform of weight 1 and level N, there is some Galois representation into $\mathrm{GL}_2(\mathbb{C})$ that gives rise to it. What about the converse? Does *every* possible irreducible representation in $\mathrm{GL}_2(\mathbb{C})$ have some Hecke eigenform for it? (That is, does every finite subgroup of $\mathrm{GL}_2(\mathbb{C})$)

Question 1 (Solved)**.** *Which $2D$ Galois irreducible representations are associated with cuspidal modular eigenforms (Hecke eigenforms)?*

For the case of the characteristic zero field $\mathbb{C}$ this is just casework: Dihedral, Tetrahedral, Octahedral, and Icosahedral. All of these have been proven. The last case was much harder than the others because A_5 is not solvable.

5 The Maass Forms

We now concern ourselves with the second part of Artin's conjecture. Maass forms of weight zero and eigenvalue 1/4.

Selberg's Trace formula showed that there are an abundance of Maass eigenforms on $X(N)$ **(Note, in class I said an abundance were shown to exist near eigenvalue 1/4, but this isn't necessarily true)**. It's also the way that Maass forms were shown to exist on $X(1) = \Gamma\backslash\mathbb{H}$. The trace formula provides a "Weyl-like" law for counting the number that appear.

Boo has shown (recently) that if Artin's conjecture holds and $L(s, \rho)$ is entire on the complex plane, then the associated function

$$\phi(z) = \sum_{n=1}^{\infty} \lambda_p(n) y^{1/2} K_0(2\pi ny)\cos(2\pi nx) \tag{11}$$

is a Maass form for $X(N)$. Note K_0 is the zeroth Bessel function. The L function is related to the Maass form by a variant of the Mellin transform.

In general:

$$\psi = \sum_{n=1}^{\infty} a(n) y^{1/2} K_{it}(2\pi ny)\cos(2\pi nx) \tag{12}$$

is easily seen to satisfy $-\nabla^2\psi = (\frac{1}{4} + t^2)\psi$, but is not usually $\Gamma(N)$ invariant.

Question 2 (Not solved)**.** *Do all Maass forms of weight* 0 *and eigenvalue* 1/4 *have coefficients corresponding to some even Galois representation? Which possible types of even two dimensional irreps give rise to these Maass forms.*

So Artin's conjecture would imply that even Galois representations would give rise to Maass forms. There has been less progress on figuring out which of the possible *even* Galois representations give rise to Maass forms. In particular, we do not know if the icosahedral case $\tilde{A}_5$ appears as a Maass form (the difficulty is due in part to the non-solvability of A_5). The dihedral case holds because Artin's conjecture is true for $2D$ representations induced from 1-dimensional ones. The tetrahedral and octahedral cases have also been recently proved to have Maass forms of that type. The proof for these cases heavily uses spectral theory and appeals to the trace formula. The icosahedral case remains an open and challenging problem: a part of Artin's conjecture that has still not been tackled.

Also worth special attention is that the theorem of Deligne and Serre for weight 1 Hecke eigenforms does not have an analogue to Maass forms. That is: Given a Maass form, we do not know if we can construct a corresponding rational field extension with Galois group that allows a $2D$ representation reproducing the form's Fourier coefficients. It is widely believed to be true.

6 Eigenvalues away from 1/4

Question 3 (Likely No)**.** *Do the Maass forms that do not have eigenvalue* 1/4 *have interesting properties in number theory?*

For eigenvalues away from 1/4, there are no known links between those Maass forms and Galois theory. If we were to write down a Maass form as in equation (12), with $t \neq 0$, then its Laplacian eigenvalue would be away from 1/4. It is widely believed that the eigenvalues of the Hecke operators on Maass forms of eigenvalue $> 1/4$ are trascendental.

Sarnak has shown this for the special case of when the coefficeients arise from a Dihedral representation. That is, if we let $a(n)$ in Equation (12) arise from dihedral Galois representations but let $t \neq 0$, then the Hecke eigenvalues are irrational for all operators. This is in contrast to the case when $t = 0$ at 1/4, where the Hecke eigenvalues are always algebraic.

Little else has been shown, but by numerical computations (Booker et al) show that the Hecke eigenvalues of general Maass forms away from 1/4 are not roots of polynomials of degree ≤ 10 with reasonably sized coefficients.

7 Summary

Hecke eigenforms of weight 1 with a character twist have coefficients stemming from Galois reprsentations in $\mathrm{GL}_2(\mathbb{C})$, as has been shown by Serre and Deligne. A corresponding statement for Hecke eigenforms of weight 0 has not been proven as of yet. Every one of the possible finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ (corresponding to dihedral D_{2n}, tetrahedral A_4, octahedral S_4, icosahedral A_5 in PGL_2) has been shown to have corresponding Hecke eigenforms.

It has been shown that the dihedral, tetrahedral, and octahedral cases of represenations give rise to Maass forms, but the existence of the Icosahedral case has evaded all attempts of proof. In general, and perhaps not surprisingly, much less is known about these non-holomorphic Maass forms than their cuspidal relatives.

In particular, for Maass forms away from eigenvalue 1/4, no connection with Galois theory has been shown, and it is widely believed (from numerical experiments) that these forms do not have any properties related to algebraic numbers.

8 Acknowledgements

I'd like to first thank Michael Magee for providing this course and expanding my knowledge of modular forms, spectral theory, and the uses of represention theory to understand the spectrum of the Laplacian on hyperbolic surfaces. Also, I'd like to thank Stanislav Atanasov for introducing me to the rudimentary basics of algebraic number theory necessary to understand the literature. Lastly I want to thank Vesselin Dimitrov for valuable evening conversation, introducing me to the advanced topics. I'd especially like to thank him for his patience in introducing this new material to a student who had not previously encountered it.

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